An Exce-L-ent Algorithm

The L-shaped 2-5-3-7 algorithm, combining efficient Singaporean and Korean procedures with divisibility rules of primes 2, 3, 5, and 7, helps students identify LCMs and GCFs.

Jae Ki Lee, Kyong Mi Choi, and Melissa McAninch

Research has proved that American students, as well as some adults, struggle with understanding fraction concepts and operations (Behr et al. 1992; NCES 2011). Having a solid understanding of this topic is important because fraction concepts are a foundation for many areas in secondary school mathematics, such as rate of change, rational expressions, functions, and equations (Son 2011). Researchers have turned to studying instructional approaches of higher-performing countries, comparing written curricula cross-nationally, with the goal of explaining differences in achievement among countries (Son and Senk 2010; Son 2011;
Watanabe 2003). They have also analyzed teacher knowledge for instruction (Son and Crespo 2009).

This article’s purpose is not to compare instructional techniques but instead to propose a new instructional technique that blends ideas from different sources. The L-shaped 2-5-3-7 algorithm combines the efficiency of an L-shaped representation found in Singaporean and Korean mathematics textbooks with a simplified procedure that uses divisibility rules of primes 2, 3, 5, and 7 from U.S. textbooks. From the elementary school to the remedial collegiate level, this representation can help students as they develop procedural skills and understanding of the factorization of numbers and operations of fractions.

Part of the problem with fraction concepts for students in a remedial program is a lack of number sense and fluency with operations. Brown and Quinn (2006) examined the challenges in rational number operations presented for students and found that more than 50 percent of participants did not understand addition or subtraction of fractions. Also found was that 48 percent did not understand multiplication of fractions. Moreover, “twenty-seven percent of students did not correctly reduce the fraction 24/36” (Brown and Quinn 2006, p. 31).

Procedural fluency is one of the five strands of mathematical proficiency. It should not be forgotten as teachers strive toward students’ overall understanding. To be procedurally fluent, a student must be able to complete mathematical procedures “flexibly, accurately, efficiently, and appropriately” (NRC 2001, p.116). The instructional hierarchy model proposed by Haring and Eaton (1978) also recognizes the need in mathematics for mastery of computational fluency before more advanced understanding and reasoning, such as that used in problem solving, can occur.

Through our mathematics teaching experiences from middle schools and
high schools to community colleges and beyond, we noticed that students struggled with fraction concepts despite myriad instructional methods used by different instructors. We also realized that one strategy did not work for all students, and we were encouraged to develop an alternative approach to help more students understand the areas where difficulties appeared.

After meeting with students and volunteering at the mathematics learning center, we realized that the common textbook approaches that fail students are verbose and contain too many steps. Students, especially those who struggle with learning mathematics, simply tried to memorize formulas and procedures to pass the test. In so doing, they did not gain a solid understanding of the concepts. Unfortunately, some students continued to be overwhelmed and failed the mathematics courses they were taking.

**APPROACHES USED IN U.S., KOREAN, AND SINGAPOREAN TEXTBOOKS**

**The L-Shaped Representation**

It is hard to say exactly when the L-shaped representation began, but there is historical evidence of its existence during previous centuries. For example, Quackenbos (1866) introduced it in *Practical Arithmetic* (p. 73). This book introduced basic mathematical concepts, such as finding the greatest common divisor (currently called the greatest common factor), division rules, and finding the least common multiple using an L-shaped representation.

**Prime Factorization Trees**

Many U.S. school mathematics textbooks, as well as Korean and Singaporean textbooks, use prime factorization concepts to teach equivalent fractions. Many textbooks, found for use at the elementary school level and those for remedial college audiences, introduce factors, multiples, and prime numbers before fraction concepts. These resources prepare students to use relevant properties and concepts of prime numbers and factorization when dealing with fractions and their operations (see, for example, Bennett et al. 2007; Charles, Crown, and Fennell 2004). Finding the greatest common factor (GCF) and least common multiple (LCM) are two topics that are closely related to fraction operations and prime factorization.

The tree diagram approach is one method that U.S. textbooks frequently use in secondary schools and community colleges to teach prime factorization. **Figure 1** displays the product of factors of a number in an organized manner. This organization is also useful for factoring large numbers because it fosters procedural flexibility by offering students choices regarding decomposition into factors. Students can capitalize on certain familiar multiplication facts and build on new ones. For example, many ways are possible to find two factors whose product is 5000, such as $50 \times 100$, $10 \times 500$, $25 \times 200$, and so forth. With various ways to decompose a number, students can flexibly explore a number’s compositions. In turn, a student with solid understanding of a composite number that can be factored in multiple ways would use this method, whereas a student having a hard time finding what number would be a factor of a given composite number might not.

![Fig. 1](image1.png) The tree diagram approach is an oft-used prime factorization method found in U.S. textbooks.

**Write the prime factorization of each number.**

- **280**
  - Write 280 as the product of two factors.
  - Continue factoring until all factors are prime.

The prime factorization of 280 is $2 \cdot 2 \cdot 2 \cdot 5 \cdot 7$, or $2^3 \cdot 5 \cdot 7$.

![Fig. 2](image2.png) This representation compares the tree diagram approach (left) to the L-shaped approach (right) to find prime factorizations.

Prime factorization of 60:

$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$
Connecting the L-Shaped Representation to Trees

The L-shaped algorithm, introduced in some Korean mathematics textbooks, provides a method of factoring. To factor 60, first write the number 60 within the L (see fig. 2). Then divide it by a prime factor, such as 2, 3, or 5, until the quotient turns out to be a prime number. The tree diagram and L-shaped representations are used with a similar procedure. Thus, any L-shaped representation can be converted into a tree representation, whereas a tree representation with more than one branching at a level cannot be converted directly into an L-shaped representation.

Figure 3 presents examples of finding the GCF and LCM introduced in a Korean middle school textbook. These examples use the same L-shaped approach to identify prime factors introduced in figure 2. The numbers at the left of the L are common factors of two given numbers. The GCF that is the multiple of all common factors of 12 and 48 is listed on the left of the L: $2 \times 2 \times 3 = 12$. The two numbers (1 and 4) in the end have only 1 as a common factor. Therefore, multiplying all factors including 1 and 4 gives the LCM: $2 \times 2 \times 3 \times 1 \times 4 = 48$. Note that the tree diagram has to be constructed for each number, but the L-shaped representation allows factoring more than one number simultaneously with an organized display of common factors. The purpose of using this representation with multiple numbers is to find the GCF or the LCM, not to find prime factorizations of the three composite numbers. For some numbers that are relatively prime, this algorithm does not yield any factor other than 1, which indicates that the given numbers are relatively prime.

One benefit of this representation is that the factors to be multiplied to find the LCM and GCF are displayed in a way that is more visually meaningful to students. By definition, the LCM of two numbers is the smallest number that is a multiple of both; in other words, the LCM is the multiple of numbers on the left of the L, which yields the GCF, and the factors located below the L. Examples are provided in figure 3, where the GCF of 60 and 100 is $2 \times 2 \times 5 = 20$ and the LCM is the multiplication of the GCF and the two remaining factors, 3 and 5: $(2 \times 2 \times 5) \times 3 \times 5 = 300$.

A drawback to the tree diagram representation is that students may struggle deciding which factors to multiply to find the LCM or GCF. This difficulty occurs because students make a common mistake, such as multiplying the same factors twice. Although the L shape is cognitively challenging because students have to find each factor that is simultaneously common to all the numbers, especially when there are common primes like 7 and 11, a systematic way to find all the common factors will make the process more efficient.

The 2-5-3-7 Sequence

All composite numbers between 1 and 120 are a multiple of at least one of the four primes 2, 5, 3, and 7. However, it could be a case that some remedial mathematics learners have a limited ability regarding number sense, which causes them to struggle with where to begin to simplify a fraction or find a prime factor of a number. Some students learn procedures to factor or to simplify a fraction but fail to retain such procedures. In turn, they do not gain a conceptual understanding with these procedures. If we scaffold this knowledge using the first four prime numbers (2, 3, 5, and 7), students can use them as a step toward being able to describe any number’s prime factorization. They can also simplify larger numbers by starting with these four primes.

Identifying the first four primes is the premise for the 2-5-3-7 algorithm because these numbers are fairly easy for students to work with. Note that instead of being represented in numerical order, the numbers 3 and 5 are switched in this algorithm because students tend to be more familiar with the divisibility rule for 5 than 3. The divisibility rules (see table 1) could be helpful for students who do not have solid number sense to understand and find the reason why the first four prime numbers can be or cannot be a prime factor of a number.

Although the 2-5-3-7 sequence is a simpler guiding approach to solve problems involving factorization of numbers, the algorithm is limited to the evaluations of relatively smaller numbers. For any composite number...
up to 120, the 2-5-3-7 algorithm is applicable. Among composite numbers larger than 120, some have a smallest prime factor larger than 7. However, only 61 composite numbers less than 1000 are exceptions to the algorithm.

Once students have developed an initial understanding of factorization with simpler, smaller numbers, students would be expected to generalize and extend the algorithm with any complex, larger number. As students start to understand and become fluent in using basic algorithms, factorization techniques using prime numbers larger than 7 should be addressed, and the 2-5-3-7 sequence can be expanded.

THE L-SHAPED 2-5-3-7 STRATEGY AND APPLICATIONS

The L-shaped 2-5-3-7 algorithm combines the L shape with the 2-5-3-7 sequence. The aim is to scaffold and create an efficient strategy that is easy for students to use in solving problems with fractions. This algorithm provides a simple graphic organizer and easier steps to follow for fraction operations that neither the L-shaped representation nor the 2-5-3-7 sequence could provide alone. We propose the L-shaped 2-5-3-7 algorithm as an efficient way to obtain the GCF and LCM and to construct prime factorizations to the point to procedural fluency. The procedural fluency resulting from the use of this algorithm will provide the foundation for work with factorization and fraction operations.

GCF and LCM

As an example, to find the GCF and LCM of two numbers in an L shape, write the two numbers 294 and 210 as shown in figure 4.

GCF = 2 × 3 × 7 = 42
LCM = 2 × 3 × 7 × 7 × 5 = 1470

Table 1 For different factors, different questions can be asked about divisibility.

<table>
<thead>
<tr>
<th>A Number Is Divisible by (No. at Left)</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>• 138; yes  • 99; no</td>
</tr>
<tr>
<td>5</td>
<td>• 670, 2115; yes  • 521; no</td>
</tr>
<tr>
<td>3</td>
<td>• 456 (4 + 5 + 6 = 15, and 15 + 3 = 5); yes  • 127 (1 + 2 + 7 = 10, and 10 + 3 = 3 1/3); no</td>
</tr>
<tr>
<td>7</td>
<td>• 602 (double 2 is 4, 60 – 4 = 56, and 56 ÷ 7 = 8); yes  • 345 (double 5 is 10, 34 – 10 = 24, and 24 ÷ 7 = 3 3/7); no</td>
</tr>
</tbody>
</table>

skip 5 on the left of the L because we are looking for a common factor, not a factor of one number.

• **Divisibility by 3:** Both 294 and 210 are multiples of 3 because the sums of all digits of each number (1 + 4 + 7 = 12 and 1 + 0 + 5 = 6) are multiples of 3: Write 3 on the left of the L and indicate the quotients of the two numbers when divided by 3 below the L. These numbers are 49 and 35. Are the two quotients still divisible by 3? No.

• **Divisibility by 7:** We find that the two numbers (49 and 35) are divisible by 7 by applying our knowledge of the 7 times table, and the quotients are 7 and 5, respectively. Since 7 and 5 are relatively prime, we stop factoring at this point.

Simplifying a Fraction

Let’s take a look at an example of simplifying a fraction, specifically, 24/36 (see fig 5). To start, write the fraction in L, and write 2-5-3-7 on
the left of the L, vertically. Since 2 and 3 at the bottom of the L are prime to each other, and the order of the numerator and the denominator are set in the beginning, the simplified form of 24/36 is 2/3. In addition, each row in this representation shows a fraction equivalent to the original fraction:

\[
\frac{24}{36} = \frac{12}{18} = \frac{6}{9} = \frac{2}{3}
\]

This example gives teachers and students a chance to discuss and reinforce the concept of equivalent fractions using the L-shaped 2-5-3-7 algorithm.

**Adding and Subtracting Fractions**

For fractions with a same denominator, the procedure for adding or subtracting the fractions is simple. However, when denominators are different, one extra step is required to find a common denominator for the fractions. This is often when students experience difficulties and decide to stop working with fractions. If there is an algorithm that provides an easy procedure to follow for finding common denominators, those students who struggle could improve their performance on problems that involve addition and subtraction of fractions. One of the benefits of applying this algorithm to those sorts of problems is that it shows the common factors as well as the multiplier needed to convert to equivalent fractions using common denominators. Another example, which provides a solution to

\[
\frac{1}{6} + \frac{1}{8}
\]

is shown in figure 6. The solution to the subtraction example

\[
\frac{5}{12} - \frac{7}{20}
\]

is shown in figure 7.

![Fig. 5 An example shows the simplification of the fraction 24/36 using the L-shaped 2-5-3-7 algorithm.](image)

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 2</td>
<td>24 / 36</td>
</tr>
<tr>
<td>5 2</td>
<td>12 / 18</td>
</tr>
<tr>
<td>3 3</td>
<td>6 / 9</td>
</tr>
<tr>
<td>7 2</td>
<td>2 / 3</td>
</tr>
</tbody>
</table>

24 and 36 are both even and divisible by 2.
12 and 18 are both even and divisible by 2.
2 and 5 are not prime factors of both numbers, but 3 is a factor of both numbers.

![Fig. 6 Adding fractions using this efficient algorithm requires two steps.](image)

**Step 1:** The L-shaped 2-5-3-7 algorithm tells us that the LCD will be 24.

**Step 2:** The diagram can also show what each denominator needs to be multiplied by to get 24: 6 × 4 = 24 and 8 × 3 = 24.

\[
\frac{1}{6} + \frac{1}{8} = \frac{7}{24}
\]

THE ALGORITHM PROMOTES FLEXIBILITY

Fractions is one area of school mathematics that has generated much research in students’ learning because of the topic’s significant utility and application in other areas and students’ difficulty with it (Wu 2009). Introducing prime factorization with the tree diagram method and referring again to the same approach to solve problems involving fractions are the usual routes taken by many U.S. mathematics textbooks, from elementary school to college. However, the fact that students continue to have difficulty understanding fractions made us realize that another approach might help when teaching fractions, preferably methods originating from places where students perform well in the area of concern.
Learning and using the L-shaped 2-5-3-7 algorithm can give a sufficient start for students who have struggled with the basic mathematics skills pertaining to fractions. It also provides an initial understanding of factorization. Although this article concentrated on using a combination of the L-shaped representation with the 2-5-3-7 sequence, the two parts of this strategy can be used separately as the teacher decides the best for his or her students. Students will gain in-depth understanding of basic skills with fractions and will be more likely to advance to higher-level mathematics courses.

Step 1: The L-shaped 2-5-3-7 algorithm tells us that the LCD will be 60.

Step 2: The diagram can also show what each denominator needs to be multiplied by to get 60: 12 × 5 = 60 and 20 × 3 = 60.

\[
\begin{align*}
\frac{5}{12} & = \frac{25}{60} \\
\frac{7}{20} & = \frac{21}{60}
\end{align*}
\]

Step 3: Check to see that 4/60 is simplified. The L-shaped 2-5-3-7 algorithm shows that our answer is 1/15.

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

Teacher needs to be mindful of the purpose for each instructional strategy and use the tool that best suits their purposes. Every tool has its benefits and drawbacks, so each must be used in a way that best fits specific goals for learning and instruction. For example, the tree diagram method is more flexible with giving students options for choosing factors, whereas the L-shaped 2-5-3-7 algorithm gives a more systematic approach. Flexibility may be more desirable when a student is exploring factorization and equivalent fractions, learning why different factors produce the same product or why two fractions are equivalent.

However, systematic methods may work better for students who need improvement with fluency as they develop speed and accuracy with solving problems (knowing how to solve problems involving factorization of numbers or operations with fractions). As discussed earlier, both conceptual and procedural knowledge are important for learning, and teaching tools may serve as outlets for achieving both kinds of knowledge in different ways.

Exploring alternative approaches in addition to those the textbook demonstrates will give teachers more flexibility in teaching mathematical concepts. This becomes especially helpful when students repeatedly report difficulties in learning the concepts taught with a particular method.

**BIBLIOGRAPHY**


NCTM Needs You!

Call for Nominations 2013 Board of Directors Election

Each year, NCTM’s Board of Directors makes important decisions that set the direction for the Council and mathematics education. The Board needs a broad representation of NCTM membership to benefit its discussions, inquiries, and decisions. In 2013, at least one high school teacher must be elected to ensure the balanced representation required by the bylaws.

NCTM has among its members many talented, energetic individuals who are qualified to assume leadership roles in the Council. The Nominations and Elections Committee needs your help in identifying these members by nominating them for Board Director positions.

Do you know someone who would bring valuable experience, perspective, and judgment to the Board? Nominate that person!

Would you be willing to serve on the NCTM Board of Directors? You can have a great impact on mathematics education at the national level.

For qualifications and responsibilities of NCTM directors and officers, school incentives for Board service, or to submit the names of nominees to the Nominations and Election Committee by completing the online nominations form go to: www.nctm.org/nominations. The chair of the committee will invite the nominee to complete an application after the first of the year.