SUPPOSE THAT IN YOUR CLASSROOM, AS part of an algebraic reasoning unit, students were asked to find a general rule for the Beam Design problem in figure 1. Examine the rules and explanations below to determine those that you would view as acceptable and those that you believe would be unacceptable. Also consider how you would respond to students who provided such explanations.

**Student 1's explanation:** Take the length of the beam and double it, then add that amount to the length of the beam doubled, minus 1. For example, for a beam of length 20, take $20 \times 2 = 40$, and $20 \times 2 - 1 = 39$. Add those together, $40 + 39 = 79$. [This rule could be represented by $2n + (2n - 1)$, where $n$ is the length of the beam. Note that any use of formal symbols, such as $2n + (2n - 1)$, are our interpretations of the students' reasoning.]

**Student 2's explanation:** I noticed that there are groups of 4 rods. There is 1 fewer group of 4 than the length of the beam, and there are 3 extra rods [see fig. 2]. My rule is: take the length of the beam minus 1, multiply that by 4, and add 3 to that amount. [This rule could be represented as $4(n - 1) + 3$.]

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Beams are designed as a support for bridges. The beams are constructed using rods. The beam length is determined by the number of rods needed to construct the bottom of the beam. Below is a beam of length 4.

1. How many rods are needed to make a beam of length 5? Of length 8? Of length 10? Of length 20? Of length 34? Of length 76?

2. How many rods are needed for a beam of length 223?

3. Write a rule or a formula to find the number of rods needed to make a beam of any length. Explain why your rule or formula works for all cases.

Fig. 1 The Beam Design problem

Fig. 2 This diagram illustrates $4(n - 1) + 3$.

Fig. 3 This diagram explores the calculation $6 \times 4 - 1$ for a beam of length 6.

Student 3’s explanation: I pretended that there was 1 more rod on the top of the diagram on the left-hand side [represented by $4n - 1$] [see fig. 3]. So, say we picked a rod of length 6, I would take the length of the beam times 4, $6 \times 4 = 24$, then subtract 1; $24 - 1 = 23$.

Each student’s explanation contains a correct rule that generalizes across all cases in the situation (where $n$ must be 1, 2, 3, . . . etc.). However, which students adequately explain why their rules always work? Which explanations help students understand related situations?
Principles and Standards for School Mathematics (NCTM 2000) calls for middle-grades students to—

- examine patterns and structures to detect regularities;
- formulate generalizations and conjectures about observed regularities;
- evaluate conjectures;
- construct and evaluate mathematical arguments.

Tasks such as the Beam Design problem provide opportunities for students to engage in all these activities. For this task, students create generalizations—statements that apply to a variety of cases. However, constructing and evaluating arguments to justify or invalidate a generalization is a challenging task for most students. To encourage students to understand why their rules apply to many cases, the teacher must create a classroom environment in which “students learn to expect and ask for justifications and explanations from one another” (NCTM 1991, p. 58). This type of environment requires negotiating and renegotiating norms about what constitutes a valid justification.

What Constitutes a Valid Justification?

In the mathematics classroom, justification involves convincing others that a statement is valid. Students determine what constitutes a valid justification from the social interactions they experience with the teacher and other students. If discussion about what constitutes a valid justification does not occur, students often rely on the superficial aspects of an argument, such as the use of formal mathematical symbols, over the mathematical reasoning that underlies the argument (Healy and Hoyles 2000). In our research with fifth-grade students, we found that all students arrived with strong conceptions (and misconceptions) about justification from their previous experiences. Those conceptions were quite resilient, impacting their willingness to consider other students’ explanations and perspectives. We wanted to establish classroom norms that encouraged students to construct valid mathematical explanations for their work.

Student justifications generally fall into four broad categories (Simon and Blume 1996): (a) no justification/procedural justification, (b) empirical justification, (c) generic examples, and (d) deductive justification. When students share their rules with others, they tend to state their rules without describing why they believe they generalize to all cases. For the Beam Design problem, student 1 explains how to determine the number of rods for a beam of any length, but it is unclear how this rule was derived or why this rule works. Thus, we classify student 1’s explanation as no justification/procedural justification. When students provide such an explanation, we want to develop the classroom norm that further explanation is necessary regarding how their rule relates to and models the problem situation. Through classroom discussions about the limitations of providing a rule without explanation, students recognize the importance of providing further explanation.

If student 1 extends his or her justification by testing several cases, an empirical justification could be provided. Students often state that a generalization is true after testing only a few instances. Such reasoning certainly adds some reassurance that the rule could apply to beams of other lengths. However, many students express concern with this type of justification (Lannin 2003), noting that some uncertainty exists as to whether we can apply this rule to other beam lengths. Another variation of empirical justification occurs when students test a unique length beam. For example, some students choose a beam of an arbitrary length, say 17; apply their rule (in the case of student 1: $17 \times 2 + 17 \times 2 - 1 = 67$); and draw a picture to confirm that their rule does provide a correct result. Again, this provides some reassurance, but some uncertainty remains that the rule applies across all values of $n$. Students should recognize that this type of justification is mathematically unacceptable for justifying a general rule and seek to understand why the rule provides correct values.

Student 3’s response is a generic example. She refers to a beam of length 6, describing how each component of her rule relates to how her rule determines the number of rods. She notes the 6 groups of 4 rods in her diagram, then explains that to determine the total number of rods, we must subtract away the “extra” rod on the left side of the diagram. In this case, a particular example is used to communicate generality across cases. Such an explanation could be viewed as general, provided that other students recognize how this rule applies to beams of different lengths. Teachers need to question how student 3’s rule can be applied to other cases so that students bring out the general nature of the explanation.

Student 2’s response represents a deductive justification—a justification that provides a general ar-
argument that clearly explains why the rule applies to all cases of the situation. Student 2 describes a general way to count the number of groups of 4 rods, noting that this amount is always 1 fewer than the length of the beam \((n - 1)\). Therefore, the expression \(4(n - 1)\) counts the number of rods in these groups of 4. Then, the extra 3 rods can be added to the expression to determine the total number of rods for a beam of any length. This justification does not describe a particular instance as does the generic example. Instead it describes a general relationship that holds true for any case.

**Why Is Justification Important?**

Too often, students emerge from algebra courses with an ability to manipulate algebraic symbols but an inability to model situations algebraically and apply the concept of variable to appropriate situations. Discussions regarding justification allow students to (a) observe how a rule applies across various cases, (b) construct generalizations to related situations, and (c) reflect on their own reasoning regarding the viability of their rules.

A sound explanation describes why a generalization properly models a situation for all values of the variable. For the beam design problem, such an explanation should describe a general relationship that can be observed in each instance. This type of justification is difficult to provide for student 1’s \(2n + (2n - 1)\) rule. We encourage the reader to consider how this rule relates to the diagram for constructing a beam of length \(n\). However, both student 2 and student 3 refer to an important invariant relationship in this situation, that groups of 4 rods can be counted as the beam’s length increases.

A second reason for emphasizing justification is that it develops an understanding that permits others to construct generalizations for similar situations. For example, consider figure 4, the poster problem. Using a similar line of reasoning to student 2 for the beam design problem, a student may notice that one poster requires 6 tacks and that each additional poster requires 3 more tacks. Thus, the total number of tacks could be determined by the expression \(6 + 3(n - 1)\), where \(n\) is the number of posters. Understanding why the generalization properly modeled the situation for the beam design problem allows students to extend their ideas to a seemingly unrelated situation, the poster problem.

Finally, students need to monitor their own thinking as they construct generalizations. When students create justifications, they naturally think about the meaning that they have established for their generalizations. They question what the various parts of their rules represent (e.g., why am I multiplying by \(4\)?). As such, teachers should take advantage of students’ natural reasoning so that connecting rules to meaning becomes the norm.

**Implications for Instruction**

As we engaged students in discussions, the students who had difficulty constructing justifications fell into two groups. One group of students did not see the general relationships that existed as they examined particular cases. They viewed each calculation as a new situation, failing to recognize that some conditions stayed the same across cases. Another group of students assumed that any rule was valid provided that it generated the correct values for a small number of cases. We employed two means to help students move beyond these initial conceptions: (a) creating tasks that focus student discussion on the validity and power of their justifications, and (b) encouraging discussion regarding how students’ justifications related to their generalizations. These two strategies are intertwined; as students generalize various situations, their conceptions emerge in classroom discussions about what constitutes a valid justification. Tasks such as the
Beam Design problem and the Poster problem are excellent initial tasks to bring out these conceptions.

After students examined and discussed a few tasks, we provided a few fictitious student justifications to encourage further examination of what constitutes a valid justification. The example in figure 5 for the Poster problem is such a task. In this example, Mollie and Sam provide two variations of an empirical justification, whereas Rachel provides a deductive justification. Rather than focusing on the creation of a rule, this task encourages discussion about the validity and power of various explanations. As discussed previously, some students feel strongly about the need to provide examples, whereas other students recognize the validity of Rachel’s general argument. This type of task will convince some, but not all students, that a general argument has more mathematical power than testing particular cases.

To strive toward the goal of presenting mathematically valid justifications, we questioned students about the general nature of their rules. We consistently asked questions such as these:

- What is changing in this situation?
- What stays the same?
- How does your rule relate to the problem situation?
- How do you know that your rule will work for 105 posters (or some other relatively large number)?
- Will your rule always work (for all appropriate values in the situation)?
- How do you know your rule will always work?

Asking these questions and encouraging students to ask each other these questions established the norm for constructing and evaluating mathematical arguments.

In addition, during whole-class discussions we wanted students to compare their justifications and assess the validity of each other’s arguments. To encourage this, we asked the following questions:

- How does Justin’s explanation relate to Elizabeth’s explanation?
- Does Elizabeth’s explanation describe why her rule always works?
- Can you explain why Justin’s rule will or will not always work?
- How does this problem relate to other problems
that we have done?

• Which of the explanations from the previous task helped you write a rule for this situation?

In doing so, we wanted whole-class discussion to be not only a time to share strategies but also an opportunity for students to assess the validity of other students’ arguments.

Conclusion

ALGEBRAIC THINKING IN THE MIDDLE grades involves constructing generalizations, moving beyond the focus of specific calculations in elementary school. This creates exciting opportunities to examine the validity of general arguments. Creating rich mathematical tasks and asking questions related to justification encourage students to examine what constitutes a valid justification for a general statement. Such reasoning bridges the mathematical activity that occurs at the elementary level and the reasoning that is necessary at the high school level.

Bibliography


